Math 451: Introduction to General Topology
Lecture 1

Basic (naive) set theory.

Set operations. Let X be a set. For a subset $A \subseteq X$, we write A^c or $X \setminus A$ for the complement of A inside X, i.e. $A^c := \{x \in X : x \notin A\}$.

For a set $\mathcal{S}$ of sets, we denote by $\bigcup \mathcal{S}$ the union of all the sets in $\mathcal{S}$, i.e.

$$\bigcup \mathcal{S} := \{x : \exists A \in \mathcal{S} \ x \in A\}.$$

In particular, if $\mathcal{S} := \{A, B\}$ then $\bigcup \mathcal{S} = A \cup B$. In other words,

$$\bigcup \mathcal{S} = \bigcup_{A \in \mathcal{S}} A.$$

Another example: if $\mathcal{S} = \{A_n : n \in \mathbb{N}\}$ then $\bigcup \mathcal{S} = \bigcup_{n \in \mathbb{N}} A_n$.

Similarly, for a set $\mathcal{S}$ of sets, we denote by $\bigcap \mathcal{S}$ the intersection of all sets in $\mathcal{S}$:

$$\bigcap \mathcal{S} := \{x : \forall A \in \mathcal{S} \ x \in A\}.$$

For a set X, its powerset $\mathcal{P}(X)$ is the set of all subsets of X, including the $\emptyset$. For example, if X has 3 elements, then $\mathcal{P}(X)$ has 2^3 elements.

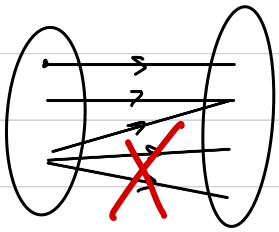
For sets X, Y their (Cartesian) product is the set

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

For example if X has 2 elements and Y has 3 elements, then $X \times Y$ has 6.

All these sets exist by the set theory axioms (of the Zermelo-Fraenkel set theory).

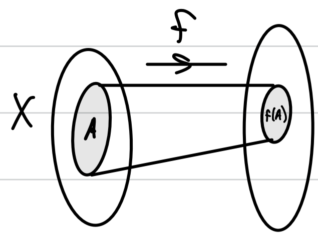
Functions. A function f from a set X to a set Y is a "set of arrows" from X to Y satisfying that for each $x \in X$ there is exactly one outgoing arrow from x to an element in Y. Formally, f is a subset of $X \times Y$ such that $\forall x \in X$ there is a unique $y \in Y$ with $(x,y) \in f$.



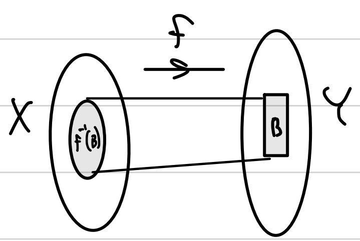
However, instead of writing $(x,y) \in f$ we write $f(x) = y$. We denote a function f from X to Y by $f : X \to Y$.

For a function $f : X \to Y$, and sets $A \subseteq X$, $B \subseteq Y$, we denote by

- $f(A) := \{f(a) : a \in A\} = \{y \in Y : \exists a \in A \; f(a) = y\}$, and call this set the f-image of A.



- $f^{-1}(B) := \{x \in X : f(x) \in B\}$, and call this set the f-preimage of B.



Note that $f(f^{-1}(B)) \subseteq B$ and $f^{-1}(f(A)) \supseteq A$, and equalities hold if f is, respectively, surjective and injective.

Btw, f is said to be injective if each $y \in Y$ has ≤ 1 f-preimage, i.e. $f^{-1}(\{y\})$ has ≤ 1 element; equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$ for all $x_1, x_2 \in X$. On the other hand f is said to be surjective iff every $y \in Y$ has an f-preimage, i.e. $f^{-1}(\{y\}) \neq \emptyset$. Finally, f is bijective if it's both injective and surjective.

For $f : X \to Y$ and $g : Y \to Z$, define $g \circ f : X \to Z$ by $g \circ f(x) := g(f(x))$, and call this the composition of f and g. Composition is associative, i.e. if $f: X \to Y$, $g: Y \to Z$, $h: Z \to W$ then $h \circ (g \circ f) = (h \circ g) \circ f$.

For $f: X \to Y$, we call a function $g: Y \to X$ its

- left inverse if $g \circ f: X \to X$ is the identity, i.e. $g \circ f(x) = x$ for all $x \in X$.
- right inverse if $f \circ g: Y \to Y$ is the identity on Y.
- inverse (two-sided) if it is both a left and a right inverse, i.e.

$$\forall x \in X \quad g \circ f(x) = x \quad \text{and} \quad \forall y \in Y \quad f \circ g(y) = y.$$

Remark. Left and right, if exist, may not be unique, but if an inverse exists, it is unique and we denote it f^{-1}. To see the uniqueness, observe that if g is a right inverse and h is a left inverse, then $g = id_X \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ id_Y = h$.

Prop. Let $f: X \to Y$, where $X \neq \emptyset$.

(a) f admits an inverse $\Leftrightarrow$ f is bijective.

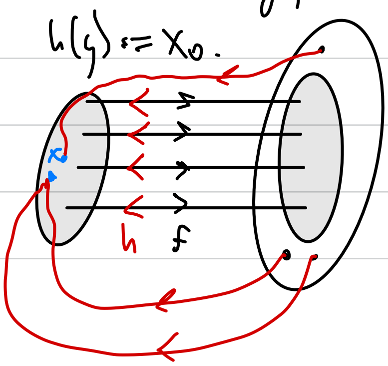
(b) f admits a left inverse $\Leftrightarrow$ f is injective.

(c) f admits a right inverse $\overset{(AC)}{\Leftrightarrow}$ f is surjective.

Proof. (a) Follow from (b)+(c) but also can be proved directly, left as an exercise.

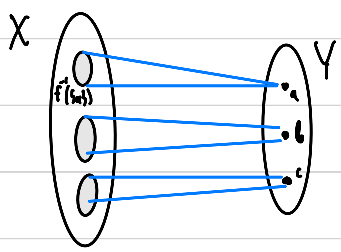
(b) $\Rightarrow$: Let $g: Y \to X$ be a left inverse of f, i.e. $g \circ f = id_X$. Then for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = g \circ f(x_1) = g \circ f(x_2) = x_2$.

$\Leftarrow$: If f is injective, then for each $y \in f(X)$, there is exactly one $x \in X$ with $f(x) = y$, denote $h(y) := x$. Fix $x_0 \in X$ and define, for each $y \in Y \setminus f(X)$, $h(y) := x_0$. Clearly, $h \circ f(x) = x$ for all $x \in X$, so h is a left inverse of f.



(c) $\Rightarrow$. Let g be a right inverse of f, i.e. $f \circ g(y) = y$ for all $y \in Y$. Fix $y \in Y$. Then $g(y) \in X$ and $f(g(y)) = f \circ g(y) = y$, so $y \in f(X)$, hence f is surjective.

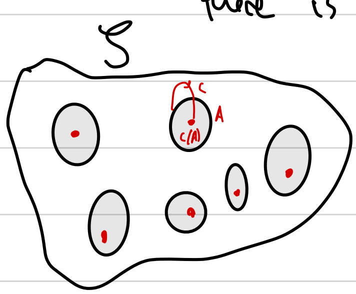
$\Leftarrow$ (AC). Suppose f is surjective. Then define $g: Y \to X$ by mapping each $y \in Y$ to some element of $f^{-1}(\{y\})$.



Then indeed, for all $y \in Y$, $f \circ g(y) = f(g(y)) = y$, so g is a right inverse of f.

There is an issue with the definition of g: g is making an arbitrary choice of an element of $f^{-1}(\{y\})$ for $y \in Y$ and the existence of such a choice function has to be asserted by a separate axiom, called Axiom of Choice (AC).

Axiom of Choice: Every set $\mathcal{S}$ of nonempty sets admits a choice function, namely, there is a function $c: \mathcal{S} \to \bigcup \mathcal{S}$ such that for each $A \in \mathcal{S}$, $c(A) \in A$.



HW: Prove that AC is equivalent to every surjection admitting a right inverse.

Equinumerocity.

Sets A, B are said to be equinumerous if there is a bijection $f: A \to B$. We denote this by $A \equiv B$. Note that $A \equiv B \Leftrightarrow B \equiv A$ because inverse of a bijection $A \to B$ is a bijection $B \to A$. We also write $A \hookrightarrow B$ if there is an injection $f: A \to B$, and write $A \twoheadrightarrow B$ if there is a surjection $A \to B$.

Cantor–Schröder–Bernstein. If $A \hookrightarrow B$ and $B \hookrightarrow A$ then $A \equiv B$.